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Technical Notes

Optimization of Natural Convection in Open Vertical Ducts with Heated Cores

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Nomenclature

A_n = coefficients
a = $1 - b$ or normalized gap width between walls
B_n = coefficients
b = $1 - a$ or ratio of size of inner cylinder to that of outer cylinder
E = nondimensional total energy transport
g = gravitational acceleration, $\mathrm{m/s^2}$
j = integer
k = thermal conductivity, $\mathrm{W/m^2K}$
L = length scale, m
n = integer or normal direction
Q = nondimensional flow rate
q = heat flux, W/m
r = normalized radial coordinate
T = temperature, K
T_a = ambient temperature, K
T_i = temperature of inner cylinder, K
W = velocity scale, m/s
w = normalized axial velocity
$\bar{w}$ = axial velocity, m/s
x = normalized Cartesian coordinate
x_j = collocation points
y = normalized Cartesian coordinate
β = coefficient of thermal expansion, 1/K
β_n = $(n - 0.5)\pi$
ν = kinematic viscosity, $\mathrm{m^2/s}$

I. Introduction

NATURAL convection in vertical ducts is important in the transport of gasses and in the cooling of electronic components. Analytic solutions for the problem often exist for fully developed flow. These solutions not only serve as benchmarks for approximate or numerical methods, but also serve as bounds for maximal flow or heat transfer, especially in microgravity conditions.

For ducts with two distinct boundaries, the temperature on each boundary may be different. The solution of fully developed natural convection between parallel plates is well known [1,2]. Joshi [3] found the exact solution in an annular duct with different constant surface temperatures and El-Shaarawi and Al-Nimr [4] studied several combinations of boundary conditions. Extensions to vertical transient annular flow [5,6] and eccentric annular flow [7,8] were made. On the other hand, horizontal annular ducts were also investigated [9,10].

The purpose of the present Note is to find the optimum dimensions for maximal flow or energy transport. This aspect has not been considered by previous authors. Also, the singular behavior when the core is very small has not been investigated. Furthermore, the nested square duct is solved here for the first time.

The outer wall is kept at ambient temperature T_a, and the inner wall is under constant-temperature or constant-flux conditions. We assume that the flow is fully developed: i.e., the velocity, pressure gradient, temperature are all independent of axial position. The energy and momentum equations under Boussinesq approximation are

$$\nabla^2 T = 0, \qquad \nu\nabla^2\bar{w} + g\beta(T - T_a) = 0 \tag{1}$$

Note that under fully developed open-ended conditions, the pressure gradient is necessarily zero. If the inner wall has constant temperature T_i, normalize all lengths by a characteristic length L, normalize the velocity by $W = g\beta L^2(T_i - T_a)/\nu$, and let $\tau = (T - T_a)/(T_i - T_a)$. Then Eqs. (1) become

$$\nabla^2\tau = 0 \tag{2a}$$

$$\nabla^2 w + \tau = 0 \tag{2b}$$

The boundary conditions are $\tau = 1$ on the inner boundary, $\tau = 0$ on the outer boundary, and $w = 0$ on all walls.

If the inner wall is under constant flux q, normalize the velocity by $W = g\beta q L^3/(k\nu)$ and let $\tau = k(T - T_a)/(qL)$. Then Eqs. (2) are the same, only the boundary condition for temperature on in inner wall is replaced by $\partial\tau/\partial n = -1$, where n is the outward normal.

II. Analysis for the Annular Duct

Natural convection in an annular duct admits a closed form exact solution. Let L be the radius of the outer cylinder and let bL be that of the inner cylinder (Fig. 1a). For the constant-temperature case, the solutions are

$$\tau = \frac{\ln r}{\ln b}, \qquad w = \frac{1}{4(\ln b)^2}\{(1 - b^2)\ln r - \ln b[1 - r^2 + (r^2 - b^2)\ln r]\} \tag{3}$$

The mass flux through the annulus, normalized by WL^2, is

$$Q = 2\pi\int_b^1 wr\,\mathrm{d}r = \frac{\pi}{32(\ln b)^2}\{-\ln b[3 + 4b^2 - 7b^4 + 4b^4\ln b] - 4(1 - b^2)^2\} \tag{4}$$

The solutions for w and Q agree with those of [3,4]. The total energy transported, normalized by $WL^2(T_i - T_a)$, is

$$E = 2\pi\int_b^1 w\tau r\,\mathrm{d}r = \frac{\pi}{64(\ln b)^3}\{\ln b[5 + 16b^2 - 21b^4 + 4b^4(5 - 2\ln b)\ln b] + 8(1 - b^2)^2\} \tag{5}$$

It is obvious for very small inner radius b that Eqs. (3) show that both temperature and velocity are zero, and thus the mass flux and the energy transported are also zero. On the other hand, if the inner radius approaches the outer radius, w and τ are bounded but the area is zero. Thus, the mass flux and the energy transported are again zero. There must be an optimum inner radius such that Q or E is maximized.

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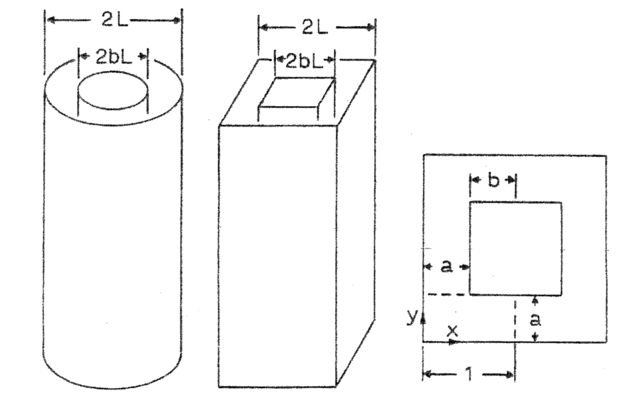

a) b) c)

Fig. 1 Sketches of a) annular duct, b) nested square duct, and c) domain of analysis for the nested square duct.

Indeed, Fig. 2 shows that maximum Q is 0.05697 at $b = 0.0948$, and maximum E is 0.01784 at $b = 0.1903$. Note also the sharp rise of the curves at small b. Asymptotic expansions of Eqs. (4) and (5) show, as $b \to 0$, the singular behavior

$$Q \sim \frac{3\pi}{32|\ln b|}, \qquad E \sim \frac{5\pi}{64(\ln b)^2} \tag{6}$$

As $b \to 1$, let $a = 1 - b$ and expand in a. Equations (4) and (5) yield

$$Q \sim \frac{\pi a^3}{12}, \qquad E \sim \frac{2\pi a^3}{45} \tag{7}$$

If the inner cylinder has constant flux, the solution is

$$\tau = -b \ln r$$
$$w = \frac{b}{4 \ln b}\{\ln b[1 - r^2 + (r^2 - b^2)\ln r] - (1 - b^2)\ln r\} \tag{8}$$

The flow rate normalized by WL^2 is

$$Q = \frac{\pi}{32 \ln b}\{b \ln b[3 + 4b^2 - 7b^4 + 4b^4 \ln b] + 4b(1 - b^2)^2\} \tag{9}$$

The forms for w and Q agree with those of [4]. The energy transport normalized by WqL^3/k is

$$E = \frac{\pi b^2}{64 \ln b}\{\ln b[5 + 16b^2 - 21b^4 + 4b^4(5 - 2\ln b)\ln b] + 8(1 - b^2)^2\} \tag{10}$$

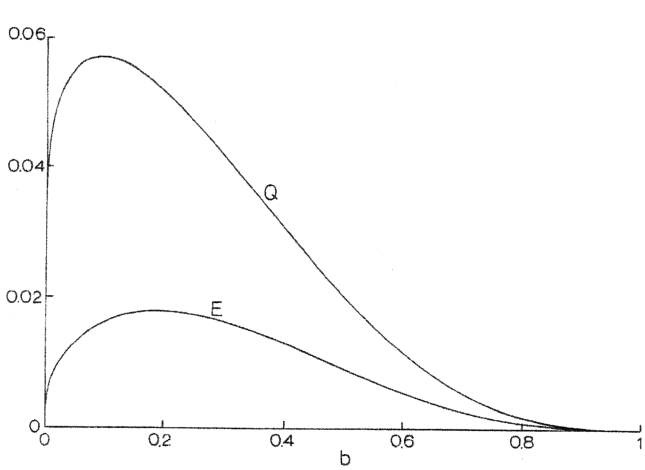

Fig. 2 Flow rate Q and energy transport E for the annular duct: inner wall with constant temperature.

Figure 3 shows maximum Q is 0.01667 at $b = 0.2095$ and maximum E is 0.002123 at $b = 0.2842$. The asymptotic expansions show for small b the rise ceases to be singular

$$Q \sim \frac{3\pi b}{32}, \qquad E \sim \frac{5\pi b^2}{64} \tag{11}$$

and for $a = 1 - b \ll 1$,

$$Q \sim \frac{\pi a^4}{12}, \qquad E \sim \frac{2\pi a^5}{45} \tag{12}$$

These asymptotic properties are used in the next section.

III. Analysis for the Nested Square Duct

Figure 1b shows a duct composed of two concentric nested square cylinders. The outer cylinder is $2L$ by $2L$ and the inner cylinder is $2bL$ by $2bL$. No exact solution exists. We shall use an eigenfunction expansion and point-match method first advocated by Wang [11] for forced flow in a nested square duct. Because of symmetry, consider the normalized representative region shown in Fig. 1c. For the constant-temperature case, the boundary conditions are

$$\tau(0, y) = 0, \qquad \frac{\partial \tau}{\partial x}(1, y) = 0, \qquad \tau(x, 0) = 0 \tag{13}$$

and on the top of the region,

$$\tau(x, a) = \tau(a, x), \qquad 0 \le x < a \tag{14}$$

$$\tau(x, a) = 1, \qquad a < x \le 1 \tag{15}$$

Note that Eq. (14) is due to the symmetry of the corner region. The solution to Eq. (2a) that satisfies Eqs. (13) is

$$\tau = \sum_{n=1}^{\infty} A_n \sin(\beta_n x)\sinh(\beta_n y) \tag{16}$$

where $\beta_n = (n - 0.5)\pi$ and A_n are coefficients. Then Eqs. (14) and (15) are satisfied by point match at N discrete points. The N unknowns A_n are then inverted. The general solution to Eq. (2b) that satisfies the boundary conditions

$$w(0, y) = 0, \qquad \frac{\partial w}{\partial x}(1, y) = 0, \qquad w(x, 0) = 0 \tag{17}$$

is

$$w = \sum_{n=0}^{\infty} \sin(\beta_n x)\left[B_n \sinh(\beta_n y) - \frac{A_n}{2\beta_n} y \cosh(\beta_n y)\right] \tag{18}$$

The last term is a particular solution. The boundary conditions are

$$w(x, a) = w(a, x), \qquad 0 \le x < a \tag{19}$$

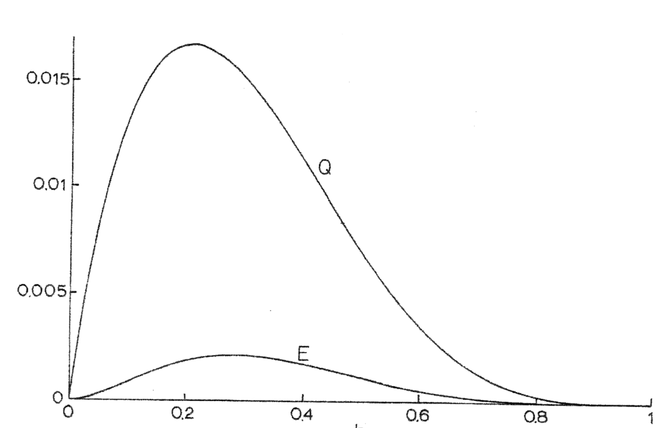

Fig. 3 Flow rate Q and energy transport E for the annular duct: inner wall with constant flux.

Table 1 Convergence of Q and E (in parenthesis)

	N					
a	10	20	30	40	50	60
0.5	0.02695 (0.01209)	0.02687 (0.01210)	0.02685 (0.01210)	0.02684	0.02683	0.02683
0.9	0.07975 (0.02234)	0.07978 (0.02251)	0.07978 (0.02263)	(0.02265)	(0.02267)	(0.02268)

$$w(x, a) = 0, \qquad a < x \le 1 \tag{20}$$

The series is truncated and point match is similarly applied, yielding the coefficients B_n. The total flow is integrated:

$$\begin{aligned} Q &= 4\int_0^a \int_0^a w \,\mathrm{d}y\,\mathrm{d}x + 8\int_a^1 \int_0^a w\,\mathrm{d}y\,\mathrm{d}x \\ &= 4\sum_{n=1}^{N} \frac{1+\cos(\beta_n a)}{\beta_n^2}\Big\{B_n[\cosh(\beta_n a) - 1] \\ &\quad - \frac{A_n}{2\beta_n^2}[\beta_n a \sinh(\beta_n a) - \cosh(\beta_n a) + 1]\Big\} \end{aligned} \tag{21}$$

The energy transport is

$$\begin{aligned} E &= 4\int_0^a \int_0^a w\tau \,\mathrm{d}y\,\mathrm{d}x + 8\int_a^1 \int_0^a w\tau\,\mathrm{d}y\,\mathrm{d}x \\ &= 2\sum_{n=1}^{N}\sum_{m=1}^{N} A_n\{4(-1)^{n+m} - \cos[(n-m)\pi a] \\ &\quad + \cos[(n+m-1)\pi a]\}[B_m J_{nm} + A_m K_{nm}] \end{aligned} \tag{22}$$

where

$$J_{nm} = \frac{\beta_m \cosh(a\beta_m)\sinh(a\beta_n) - \beta_n \cosh(a\beta_n)\sinh(a\beta_m)}{\beta_m^2 - \beta_n^2}$$

$$\begin{aligned} K_{nm} &= \frac{-1}{4\beta_m}\Big\{\frac{a\cosh[a(\beta_n+\beta_m)]}{\beta_n+\beta_m} - \frac{\sinh[a(\beta_n+\beta_m)]}{(\beta_n+\beta_m)^2} \\ &\quad + \frac{a(\beta_n-\beta_m)\cosh[a(\beta_n-\beta_m)] - \sinh[a(\beta_n-\beta_m)]}{(\beta_n-\beta_m)^2}\Big\} \end{aligned}$$

The convergence can be improved by increasing N. Table 1 shows when N is increased to about 60, both Q and E no longer change (converged).

Our matching scheme becomes inaccurate when a is very close to 1, or when $b = 1 - a \ll 0$. Since the annular duct has inverse logarithmic singularities for small b [Eq. (6)], we expect the nested square duct would have similar behavior. Indeed, when $Q|\ln b|$ and $E(\ln b)^2$ are plotted in the inset of Fig. 4, one can extrapolate the following approximate formulas:

$$Q \sim \frac{0.3 - 1.43b}{|\ln b|}, \qquad E \sim \frac{0.25 - 2.4b}{(\ln b)^2} \tag{23}$$

On the other hand, if a is very small, the region in Fig. 1c is long and slender. An approximation would be a slit of width a. The temperature is $\tau = y/a$ and the velocity is $w = (a^2 y - y^3)/(6a)$. Since the length is $2 - a$, we find

$$\begin{aligned} Q &\sim 4(2-a)\int_0^a w\,\mathrm{d}y = \frac{(2-a)a^3}{6} \\ E &\sim 4(2-a)\int_0^a w\tau\,\mathrm{d}y = \frac{4(2-a)a^3}{45} \end{aligned} \tag{24}$$

Figure 4 shows Q and E for the nested square cylinder. Maximum Q is 0.0799 at $b = 0.092$ and maximum E is 0.248 at $b = 0.180$. Our approximate formulas compare well at their respective regions of validity.

For a constant-flux condition on the inner wall, Eqs. (13) and (14) remain the same and Eq. (15) is replaced by

$$\frac{\partial \tau}{\partial y}(x, a) = 1, \qquad a < x \le 1 \tag{25}$$

Then Eq. (16) yields a set of different coefficients A_n, but Eqs. (18), (21), and (22) would retain the same forms. For small $b = 1 - a$ we plotted Q/b and E/b^2 versus b and extrapolated to zero. The results show

$$Q \sim 0.4b, \qquad E \sim 0.4b^2 \tag{26}$$

For small $a = 1 - b$ the strip approximation similarly gives

$$Q \sim \frac{(2-a)a^4}{6}, \qquad E \sim \frac{4(2-a)a^5}{45} \tag{27}$$

Figure 5 shows maximum Q is 0.0277 occurring at $b = 0.195$ and maximum E is 0.00411 occurring at $b = 0.275$.

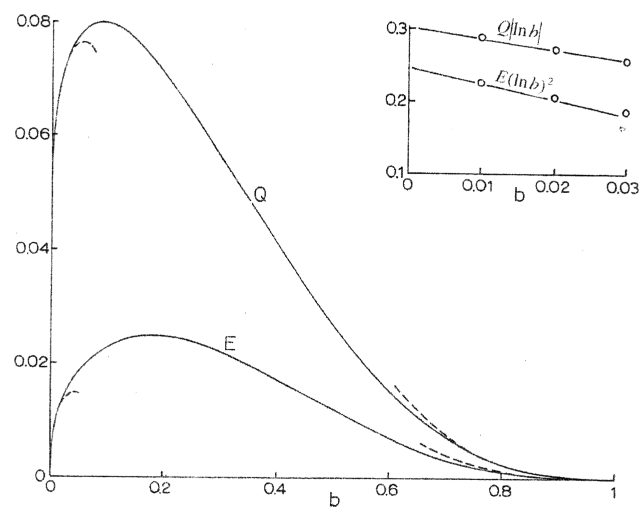

Fig. 4 Flow rate Q and energy transport E for the nested square duct: inner wall with constant temperature. Dashed lines are approximations from Eqs. (23) and (24). Inset shows the extrapolation for small b.

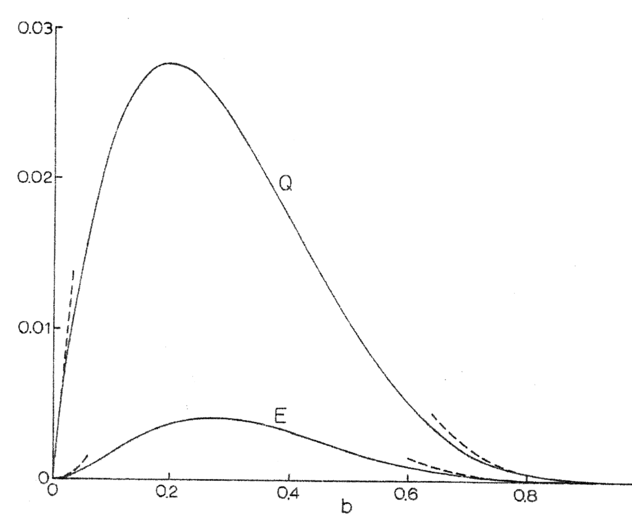

Fig. 5 Flow rate Q and energy transport E for the nested square duct: inner wall with constant flux. Dashed lines are approximations from Eqs. (26) and (27).

IV. Discussion

This Note considers the transport of mass and energy through natural convection heated by a core. The flow Q is the maximum flux possible and the energy integral E is the maximum energy that can transported (regardless of entrance effects). The optimum size of the inner cylinder (core) is determined.

Our results show that for constant-temperature heating, both Q and E rise singularly as b is increased from zero. This means that the size of the small core has a large effect. In comparison, for constant-flux heating, such rise is proportional to b and b^2 and is thus much slower.

For a parallel plate duct there is no optimum gap width: the wider the separation, the better. If the outer dimensions are the same, we find that the annular duct gives a higher Q and a higher E than those of the nested square duct, even though the latter has a larger cross-sectional area for the same b.

V. Conclusions

This Note shows, for the first time, the singular behavior for natural convection due to a small heated core. The optimum dimensions of the core sizes are also determined. Our analytic and graphical results will be useful in the design of annular and nested square ducts.

References

[1] Ostrach, S., "Laminar Flow with Body Forces," *High Speed Aerodynamics and Jet Propulsion*, edited by F. K. Moore, Vol. 4, Princeton Univ. Press, Princeton, NJ, 1964.

[2] Jaluria, Y., *Natural Convection*, Pergamon, Oxford, 1980.

[3] Joshi, H. M., "Fully Developed Natural Convection in an Isothermal Vertical Annular Duct," *International Communications in Heat and Mass Transfer*, Vol. 14, No. 6, 1987, pp. 657–664. doi:10.1016/0735-1933(87)90045-5

[4] El-Shaarawi, M. A. I., and Al-Nimr, M. A., "Fully Developed Laminar Natural Convection in Open-Ended Vertical Concentric Annuli," *International Journal of Heat and Mass Transfer*, Vol. 33, No. 9, 1990, pp. 1873–1884. doi:10.1016/0017-9310(90)90219-K

[5] Al-Nimr, M. A., "Analytical Solution for Transient Laminar Fully Developed Free-Convection in Vertical Concentric Annuli," *International Journal of Heat and Mass Transfer*, Vol. 36, No. 9, 1993, pp. 2385–2395. doi:10.1016/S0017-9310(05)80122-X

[6] Ramamurthi, K., and Ravi, K., "Transient Natural Convection in a Cylindrical Annulus," *Journal of Thermophysics and Heat Transfer*, Vol. 11, No. 4, 1997, pp. 595–597. doi:10.2514/2.6288

[7] El-Shaarawi, M. A. I., Mokheimer, E. M. A., and Abulhamayel, H. I., "Limiting Values for Free-Convection Induced Flow Rates in Vertical Eccentric Annuli with an Isothermal Boundary," *Numerical Heat Transfer, Part A, Applications*, Vol. 39, No. 6, 2001, pp. 611–630.

[8] Mokheimer, E. M. A., and El-Shaarawi, M., "Maximum Possible Induced Flow Rates in Open-Ended Vertical Eccentric Annuli with Uniform Heat Flux," *International Journal of Numerical Methods for Heat and Fluid Flow*, Vol. 15, No. 2, 2005, pp. 161–182. doi:10.1108/09615530510578438

[9] Pepper, D. W., and Cooper, R. E., "Numerical Solution of Natural Convection in Eccentric Annuli," *AIAA Journal*, Vol. 21, No. 9, 1983, pp. 1331–1337. doi:10.2514/3.8247

[10] Moukalled, F., and Archaya, S., "Natural Convection in the Annulus Between Concentric Horizontal Circular and Square Cylinders," *Journal of Thermophysics and Heat Transfer*, Vol. 10, No. 3, 1996, pp. 524–531. doi:10.2514/3.820

[11] Wang, C. Y., "Laminar Flow in Certain Ducts of Symmetric Cross Section," *Canadian Journal of Chemical Engineering*, Vol. 77, No. 1, 1999, pp. 107–112. doi:10.1002/cjce.5450770117